Transition from Uniform Motion to Stick-Slip in the Rice-Ruina Model of One and Two Blocks

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Abstract

We analyze the Rice-Ruina state and rate dependent friction model. The system consists of one or two blocks driven by springs with constant velocity on a dry, rough surface. Our discussion is limited to the creep-like motion, when the driving velocity is small. Two regimes of motion are observed: stick-slip and steady sliding. The stability of the steady sliding depends on the model parameters. Numerical and analytical results show a transition between two regimes: the system passes directly from uniform to stick-slip motion. The calculations are performed also for two driven blocks. Then, the transition has the same character and it appears at the same point.

1   Introduction

Friction between solids is a complex problem, first investigated in 1699 by Amontons [1]. In 1781, Coulomb formulated his laws, known as Amontons-Coulomb laws. Suppose that a solid block pressed to a flat surface by a normal force W, and the contact surface is of nominal area S. The static and dynamic friction coefficients μ_s and μ_d as the proportionality coefficients between the friction forces F_s and F_d and the force W. F_s is the force necessary to move a standing block. F_d is the friction force during a uniform motion. The laws state that both friction coefficients do not depend on W and S, and that $\mu_d < \mu_s$. These laws are widely used until now [1].

In recent years, interest on friction has been revived because of its possible relevance for earthquakes [2], and of its obvious relevance for technology [3]. Detailed experimental studies revealed subtle effects on nanoscopic scale, and several phenomenological models have been formulated [4, 5, 6, 7]. In particular, the creep regime is of interest, which can be considered as an intermediate stage between motion and rest. The idea is that during a very slow motion, the interface between solids preserves to some extent the information on history of contact points. For this regime, the model of Rice and Ruina [6] is considered to be appropriate [8, 9]. Some variation of this model was formulated in [7].

In these models, the physical system is a block connected to a driving mechanism by a spring. The mechanism drives the block with a constant velocity. The uniform motion of the block is not a unique solution. An alternative is the so-called stick-slip motion, when the block velocity is varying periodically from zero to some maximum, then back to zero and so on. There is an experimental

evidence that once the uniform phase loses its stability, the motion becomes oscillatory, and the amplitude of the oscillations increases gradually from zero when the control parameter departures the transition [1, 7, 9, 10]. This suggests an appearance of the supercritical Hopf bifurcation [11]. However, as it was pointed out in [12], further corrections to the Rice-Ruina equations are necessary to reproduce the gradual increase of the amplitude of the oscillations, present in the Hopf bifurcation. Without these corrections, the transition leads directly from the uniform motion to the stick-slip effect. In other words, the Rice-Ruina equations are not generic; a slight modification of the model leads to finite oscillations, and the Hopf bifurcation is observed. Examples of such modifications are described in [12, 13]; more general discussion of possible memory effects in the problem of friction can be found in [14]. However, the Rice-Ruina equations are in most cases sufficient for a convenient description of the uniform motion in the creep regime.

In realistic situations, the number of contact points between the moving surfaces is larger than one, and it depends on the normal force [3, 5, 4]. This is an indication of a gap between theory and experiment on friction. It is clear that a more appropriate and general model should contain the number of contact points varying in time, with a distribution of elastic forces between them. However, results of such a model are expected to depend on numerous parameters, which vary not only from one sample to another but also in time. In particular, the number of contact points and their elastic constants are hard to be controlled.

The aim of this paper is to compare the instability of the uniform motion for one and two blocks. In this approach, the blocks are equivalent to the contact points. Our main goal is to prove that the instability of the uniform motion occurs for the same values of the model parameters for one and two blocks. When discussing this result, we are faced with almost all above mentioned difficulties in the interpretation. In reality, the case of two contact points is probably as rare as the case of one. However, it seems to us that at least the direction is proper. We hope that our analytical and numerical results provide a basis for more extensive search.

In subsequent section we show that for one block the analytical conditions for the Hopf bifurcation [11] are not fulfilled. The calculations are supplemented with some numerical results, reported in Section III. In Section IV we prove analytically, that the transition point is the same for the case of one and two blocks. Again, the simulations support these results. Last section is devoted to final conclusions.

2   The equations of Rice and Ruina

The equations of motion can be formulated [1, 9] as follows

$$\frac{K}{W}(Vt - x) = \mu_0 + B \ln \frac{\Phi}{\Phi_0} + A \ln \frac{\dot{x}}{V_0} \tag{1}$$

$$\dot{\Phi} = 1 - \frac{\dot{x}\Phi}{D_0} \tag{2}$$

where $x - Vt$ is the block position with respect to the driving mechanism, V is the driving velocity, K is the spring constant, W is the normal force, μ_0 is

a reference value of the friction coefficient for steady sliding at some velocity V_0. During sliding, the microcontacts are refreshed, on average, after a distance D_0. The state of these microcontacts is described by the variable Φ, which interpolates between the time of stick for the block sticked and D_0/V for steady sliding. Finally, $\Phi_0 = D_0/V_0$ and A, B are unitless material constants. We note that $B > A$ in the experimental data [12].

The equations can be transformed to an autonomous form. Denoting $(x - Vt)/D_0 = \alpha$, $\Phi/\Phi_0 = \phi$, $V_0 t/D_0 = \tau$, $V/V_0 = \omega$, $\exp(-\mu_0/A) = \gamma$, $KD_0/W = \kappa$, we get unitless equations

$$\dot{\alpha} = -\omega + \gamma\phi^{-\frac{B}{A}}\exp[-\frac{\kappa}{A}\alpha] \tag{3}$$

$$\dot{\phi} = 1 - \gamma\phi^{1-\frac{B}{A}}\exp[-\frac{\kappa}{A}\alpha] \tag{4}$$

where the time derivative is over τ. These equations can be furter simplified by a change of variables: $-1/\omega \exp(-\kappa\alpha/A = x$, $\phi^{-b-1} = y$, where $b = B/A - 1$. Introducing a parameter $\mu = b - \kappa/A$, and renormalizing time once more ($\tau \to \gamma\tau$) we get the equations in more algebraic form

$$\dot{x} = (b - \mu)(mx - x^2 y) \tag{5}$$

$$\dot{y} = (1 + b)(xy^2 - \eta y^{1+\frac{1}{1+b}}) \tag{6}$$

where $m = \omega/\gamma > 0$, $\eta = 1/\gamma$. At the fixed point, where the block velocity is constant and equal to the driving velocity, $\dot{x} = 0$ and $\dot{y} = 0$. There,

$$x = \eta(\frac{\eta}{m})^b \tag{7}$$

$$y = \left(\frac{m}{\eta}\right)^{(1+b)} \tag{8}$$

The determinant of Jacobian J at the fixed point is $(b - \mu)m^2$, and the trace is μm. This means that the stability of the fixed point is lost when the control parameter μ becomes positive [11]. Near this point, $Det(J) > 0$.

In due course, only first derivatives over μ will be calculated at $\mu = 0$. Then, writing down the eigenvalues of J we can neglect terms proportional to μ^2.

$$\lambda = m(\frac{\mu}{2} \pm i\sqrt{b - \mu}) \tag{9}$$

The transformation to the Jordan form leads to new variables ξ, ψ

$$\xi = \frac{(1+b)}{m\sqrt{b}}(1 + \frac{\mu}{2b})x + \sqrt{b}(\frac{m}{\eta})^{-2(1+b)}y \tag{10}$$

$$\psi = (\frac{m}{\eta})^{-2(1+b)}y \tag{11}$$

Now, the equations of motion are

$$\dot{\xi} = \frac{m}{(1+b)}b\sqrt{b}(\frac{3}{2}\frac{\mu}{b} - 1)(\frac{m}{\eta})^{2(1+b)}(\xi - \sqrt{b}\psi)^2\psi - mb(\frac{\mu}{b} - 1)(\xi - \sqrt{b}\psi) +$$
$$+ mb(1 - \frac{\mu}{2b})(\frac{m}{\eta})^{2(1+b)}(\xi - \sqrt{b}\psi)\psi^2 - \eta\sqrt{b}(1 + b)(\frac{m}{\eta})^2\psi^{1+\frac{1}{1+b}} \tag{12}$$
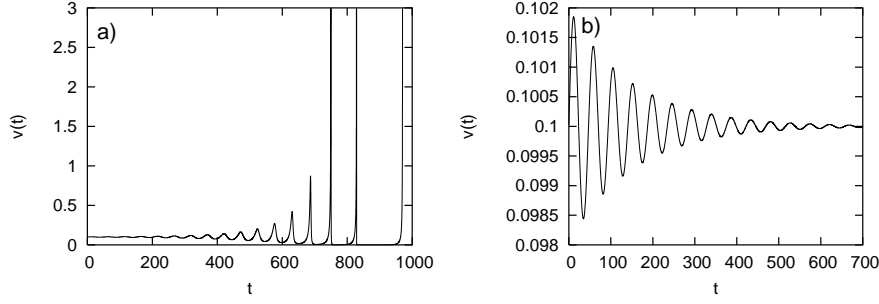
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Figure 1: Velocity of the single block versus time: (a) stick-slip (b)steady slip motion regime.

$$\dot{\psi} = m\sqrt{b}(1 - \frac{\mu}{2b})(\frac{m}{\eta})^{2(1+b)}(\xi - \sqrt{b}\psi)\psi^2 - (1+b)\eta(\frac{m}{\eta})^2\psi^{1+\frac{1}{1+b}} \qquad (13)$$

These equations can be written in short as $\dot{\xi} = f(\xi,\psi)$, $\dot{\psi} = g(\xi,\psi)$. The condition for the presence of the Hopf bifurcation [11] is

$$
\begin{aligned}
a &= \frac{1}{16}(f_{\xi\xi\xi} + g_{\xi\xi\psi} + f_{\xi\psi\psi} + g_{\psi\psi\psi}) + \\
&\quad \frac{1}{16\omega}[f_{\xi\psi}(f_{\xi\xi} + f_{\psi\psi}) - g_{\xi\psi}(g_{\xi\xi} + g_{\psi\psi}) - f_{\xi\xi}g_{\xi\xi} + f_{\psi\psi}g_{\psi\psi}] \neq 0 \quad (14)
\end{aligned}
$$

However, direct calculations for Eqns. (12,13) at the fixed point and $\mu = 0$ lead to the result $a = 0$.

As noted above, the phase of steady slip ceases to be stable when $\mu = B/A - 1 - KD_0/(WA)$ becomes positive. This can mean in particular, that the spring constant K decreases. This is in accordance with the phase diagram in the creep regime, observed experimentally [1, 7, 9, 10, 15].

3   Numerical calculations for one block

The stability of the solution of the equations of Rice and Ruina is checked numerically at both sides of the transition. We applied the Runge-Kutta method of 4-th order. The result is shown in Fig. 1. The parameters of the calculation [3] are: $B = 0.08$, $A = 0.03$, $\mu_0 = 0.4$, $V = 0.1\mu m/s$, $\phi_0 = 1.0$, $D_0 = 0.1\mu m$, $K/W = 0.054\mu m^{-1}$ for the steady slip and $K/W = 0.046\mu m^{-1}$ for the stick-slip. The initial value of Φ is slightly (0.01) different from value characteristic for uniform movement with velocity equal to driving mechanism velocity V. The initial values of block's position and velocity are equal to values characteristic for uniform movement. As we see, either the steady slip or the stick-slip is observed, without an intermediate phase with a continuous rise of the amplitude of the oscillations. Such a continuity is expected at the Hopf bifurcation.

4

4 The case of two blocks

In this case we rewrite the equations (1,2) twice with new variables x_1, x_2, Φ_1, Φ_2 and we add a coupling between the blocks by means of a new spring with constant k_2. Both driving springs have the same constants k_1. The equations are

$$\mu_0 + B\ln\frac{\Phi_1}{\Phi_0} + A\ln\frac{\dot{x_1}}{V_0} = \frac{1}{W}(k_1(vt - x_1) + k_2(x_2 - x_1)) \tag{15}$$

$$\mu_0 + B\ln\frac{\Phi_2}{\Phi_0} + A\ln\frac{\dot{x_2}}{V_0} = \frac{1}{W}(k_1(vt - x_2) + k_2(x_1 - x_2)) \tag{16}$$

$$\dot{\Phi}_1 = 1 - \frac{\dot{x_1}\Phi_1}{D_0} \tag{17}$$

$$\dot{\Phi}_2 = 1 - \frac{\dot{x_2}\Phi_2}{D_0} \tag{18}$$

Changes of variables, similar to the ones applied above, lead to autonomous dimensionless equations

$$\dot{\alpha_1} = -\omega + \phi_1^{-\frac{B}{A}}\exp(-\mu_0)\exp(\frac{k_1 D_0}{W}\alpha_1)\exp(\frac{k_2 D_0}{W}(\alpha_2 - \alpha_1)) \tag{19}$$

$$\dot{\alpha_2} = -\omega + \phi_2^{-\frac{B}{A}}\exp(-\mu_0)\exp(\frac{k_1 D_0}{W}\alpha_2)\exp(\frac{k_2 D_0}{W}(\alpha_1 - \alpha_2)) \tag{20}$$

$$\dot{\phi_1} = 1 - \gamma\phi_1^{1-\frac{B}{A}}\exp(-\mu_0)\exp(\frac{k_1 D_0}{W}\alpha_1)\exp(\frac{k_2 D_0}{W}(\alpha_2 - \alpha_1)) \tag{21}$$

$$\dot{\phi_2} = 1 - \gamma\phi_2^{1-\frac{B}{A}}\exp(-\mu_0)\exp(\frac{k_1 D_0}{W}\alpha_2)\exp(\frac{k_2 D_0}{W}(\alpha_1 - \alpha_2)) \tag{22}$$

The eigenvalues of the Jacobian at the fixed point are

$$\lambda_1 = \frac{\omega}{2AW}[-D_0 k_1 - AW + BW - \sqrt{-4AD_0 k_1 W + (D_0 k_1 + AW - BW)^2}] \tag{23}$$

$$\lambda_2 = \frac{\omega}{2AW}[-D_0 k_1 - AW + BW + \sqrt{-4AD_0 k_1 W + (D_0 k_1 + AW - BW)^2}] \tag{24}$$

$$\lambda_3 = \frac{\omega}{2AW}[-D_0 k_1 - 2D_0 k_2 - AW + BW - \sqrt{(D_0 k_1 + 2D_0 k_2 + AW - BW^2) - 4AW(D_0 k_1 + 2Dk_2)}] \tag{25}$$

$$\lambda_4 = \frac{\omega}{2AW}[-D_0 k_1 - 2D_0 k_2 - AW + BW + \sqrt{(D_0 k_1 + 2D_0 k_2 + AW - BW^2) - 4AW(D_0 k_1 + 2Dk_2)}] \tag{26}$$

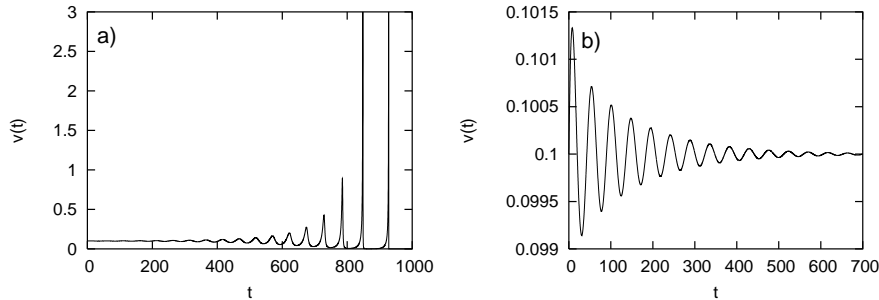Figure 2: Velocities of two blocks versus time: (a)slip-stick(b) steady slip motion regime.

Two of them, λ_1 and λ_2, change sign at the same value of k_1/W where the transition for one block occurred. At this point, λ_3 and λ_4 remain negative. This completes the proof that the transition point for two blocks coincides with the transition for one block. This is confirmed numerically, as shown in Fig. 2, for the same parameters as above, and $k_2/W = 0.05\mu m^{-1}$. In this case, the numerical plots reveal almost full synchronization of two blocks below and above the transition: the time dependences of the block velocities coincide.

5   Conclusions

We demonstrate that for two contact points, the uniform sliding ceases its stability at the same value of the elastic driving force, that for one contact point. This means, that the elastic force between the contact points, represented here as k_2, does not influence the stability of the uniform motion. A question arises, if this result could be valid for a larger number of blocks, i.e. of contact points. Such a generalization is of obvious interest; if true, it could release the limitation of the whole approach, which in this case could apply to real surfaces with multiple contact points. However, we can only state that the question still remains open.

For two blocks, our numerical results provide a demonstration of a synchronization of the stick-slip motion. However, we know that such a synchronization can depend on the initial conditions, and therefore it cannot be treated as generic citemy. This is true in particular if one tries to generalize the results for the larger number of contact points.

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